

Stability of two-dimensional solutions to the Navier-Stokes equations in cylindrical domains under Navier boundary conditions

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Abstract. The Navier-Stokes motions in a cylindrical domain with Navier boundary conditions are considered. First the existence of global regular two-dimensional solutions is proved. The solutions are such that bounded with respect to time norms are controlled by the same constant for all $t \in \mathbb{R}_+$. Assuming that the initial velocity and the external force are sufficiently close to the initial velocity and the external force of the two-dimensional solutions we prove existence of global three-dimensional solutions which remain close to the two-dimensional solutions for all time. In this way we mean stability of two-dimensional solutions. Thanks to the Navier boundary conditions the nonlinear term in two-dimensional Navier-Stokes equations does not have any influence on the form of the energy estimate. This implies that stability is proved without any structural restrictions on the external force, initial data and viscosity.

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1. Introduction

In this paper we prove stability of two-dimensional solutions in a set of three-dimensional motions of the Navier-Stokes equations in cylindrical domain $D = \Omega \times (-a, a)$, where $\Omega \subset \mathbb{R}^2$ and $L = 2a$ is the length of the cylinder. The three-dimensional motions satisfy the following initial-boundary value problem

$$\begin{aligned}
 (1.1) \quad & v_t + v \cdot \nabla v + \nu \operatorname{rot} \operatorname{rot} v + \nabla p = f \quad \text{in } D_+ \equiv D \times \mathbb{R}_+, \\
 & \operatorname{div} v = 0 \quad \text{in } D_+, \\
 & v \cdot \bar{n} = 0 \quad \text{on } S_+ = S \times \mathbb{R}_+, \\
 & \bar{n} \times \operatorname{rot} v = 0 \quad \text{on } S_+, \\
 & v|_{t=0} = v(0) \quad \text{in } D,
 \end{aligned}$$

where $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x, t) \in \mathbb{R}$ is the pressure, $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ is the external force field. By $x = (x_1, x_2, x_3)$ are denoted the Cartesian coordinates such that x_3 -axis is parallel to the cylinder and is located inside it. By the dot we denote the scalar product in \mathbb{R}^3 .

The Navier-Stokes equations (1.1)₁ follow from the formula

$$(1.2) \quad -\Delta v = \operatorname{rot} \operatorname{rot} v$$

which holds for divergence free vectors v . Finally, by ν we denote the positive viscosity coefficient.

Moreover, \bar{n} is the unit outward vector normal to S . The boundary S is split into two parts, $S = S_1 \cup S_2$, where S_1 is parallel to the x_3 -axis and S_2 is perpendicular. Additionally $S_2 = S_2(-a) \cup S_2(a)$, where $S_2(b)$ meets x_3 -axis at $x_3 = b$.

Our aim is to prove existence of global regular nonvanishing with time solutions to problem (1.1). For this we need that the external force does not converge to zero as time goes to infinity. Then to reduce restrictions on the external force we introduce the quantities

$$(1.3) \quad p' = x \cdot \oint f dx, \quad \bar{p} = p - p', \quad \bar{f} = f - \oint f dx,$$

where

$$\oint f dx = \frac{1}{|D|} \int_D f(x) dx \quad \text{and} \quad |D| = \operatorname{meas} D.$$

Then problem (1.1) takes the form

$$\begin{aligned}
(1.4) \quad & v_t + v \cdot \nabla v + \nu \operatorname{rot} \operatorname{rot} v + \nabla \bar{p} = \bar{f}, \\
& \operatorname{div} v = 0, \\
& v \cdot \bar{n} = 0, \quad \bar{n} \times \operatorname{rot} v = 0, \\
& v|_{t=0} = v(0).
\end{aligned}$$

Up to now there is not possible to prove existence of global regular solutions to problem (1.1). Therefore we restrict our considerations to show existence of global regular solutions which remain sufficiently close to two-dimensional global correspondingly regular solutions. It is well known that such two-dimensional solutions exist. Since we need a special behavior of two-dimensional solutions we show their existence in Section 3.

By two-dimensional motions we mean such solutions to (1.1) that $v = w = (w_1(x_1, x_2, t), w_2(x_1, x_2, t)) \in \mathbb{R}^2$, $p = \eta(x_1, x_2, t) \in \mathbb{R}$ and $f = h = (h_1(x_1, x_2, t), h_2(x_1, x_2, t)) \in \mathbb{R}^2$. Hence the two-dimensional motions satisfy

$$\begin{aligned}
(1.5) \quad & w_t + w \cdot \nabla w + \nu \tilde{\operatorname{rot}} \operatorname{rot}^{(2)} w + \nabla \eta = h \quad \text{in } \Omega \times \mathbb{R}_+ \equiv \Omega_+, \\
& \operatorname{div} w = 0 \quad \text{in } \Omega_+, \\
& w \cdot \bar{n} = 0 \quad \text{on } S_0 \times \mathbb{R} \equiv S_{0+}, \\
& \operatorname{rot}^{(2)} w = 0 \quad \text{on } S_{0+}, \\
& w|_{t=0} = w(0) \quad \text{in } \Omega,
\end{aligned}$$

where $S_0 = \partial\Omega$ and $\operatorname{rot}^{(2)} w = w_{2,x_1} - w_{1,x_2}$, $\tilde{\operatorname{rot}} \varphi = (\varphi_{,x_2} - \varphi_{,x_1})$. Comparing to (1.1) we see that $S_1 = S_0 \times (-a, a)$.

To examine problem (1.5) we need transformation of type (1.3) applied to the two-dimensional case. Therefore, in this case we introduce

$$(1.6) \quad \eta' = x' \cdot \int h dx', \quad \bar{\eta} = \eta - \eta', \quad \bar{h} = h - \int h dx',$$

where $x' = (x_1, x_2)$, $\int h dx' = \frac{1}{|\Omega|} \int_{\Omega} h(x') dx'$.

Using (1.6) problem (1.5) takes the form

$$\begin{aligned}
(1.7) \quad & w_t + w \cdot \nabla w + \nu \tilde{\operatorname{rot}} \operatorname{rot}^{(2)} w + \nabla \bar{\eta} = \bar{h} \quad \text{in } \Omega_+, \\
& \operatorname{div} w = 0 \quad \text{in } \Omega_+, \\
& w \cdot \bar{n} = 0, \quad \operatorname{rot}^{(2)} w = 0 \quad \text{on } S_{0+}, \\
& w|_{t=0} = w(0) \quad \text{in } \Omega.
\end{aligned}$$

Since we consider incompressible motions we can assume without any restrictions that f and h are divergence free.

To show stability of two-dimensional solutions we introduce the quantities

$$(1.8) \quad u = v - w, \quad q = \bar{p} - \bar{\eta}, \quad g = \bar{f} - \bar{h}$$

which are solutions to the problem

$$(1.9) \quad \begin{aligned} u_t + u \cdot \nabla u &= \nu \operatorname{rot}^2 u + \nabla q = -w \cdot \nabla u - u \cdot \nabla w + g && \text{in } D_+, \\ \operatorname{div} u &= 0 && \text{in } D_+, \\ u \cdot \bar{n} = 0, \quad \bar{n} \times \operatorname{rot} u &= 0 && \text{on } S_+, \\ u|_{t=0} &= u(0) && \text{in } D. \end{aligned}$$

Remark 1.1. The operator rot^2 and the boundary conditions $(1.9)_3$ hold for function v . Now we show that they are also satisfied for a solution w to problem (1.7). Using that $w = (w_1, w_2, 0)$ and $\operatorname{div} w = 0$ we have

$$\begin{aligned} \operatorname{rot}^2 w &= -\Delta^{(2)} w = - \begin{pmatrix} w_{1,x_1 x_1} + w_{1,x_2 x_2} \\ w_{2,x_1 x_1} + w_{2,x_2 x_2} \end{pmatrix} \\ &= \begin{pmatrix} (w_{1,x_1} + w_{2,x_2})_{,x_1} - (w_{1,x_1 x_1} + w_{1,x_2 x_2}) \\ (w_{1,x_1} + w_{2,x_2})_{,x_2} - (w_{2,x_1 x_1} + w_{2,x_1 x_2}) \end{pmatrix} = \begin{pmatrix} (w_{2,x_1} - w_{1,x_2})_{,x_2} \\ -(w_{2,x_1} - w_{1,x_2})_{,x_1} \end{pmatrix} \\ &= \tilde{\operatorname{rot}} \operatorname{rot}^{(2)} w, \end{aligned}$$

where $\Delta^{(2)} = \partial_{x_1}^2 + \partial_{x_2}^2$ and $\operatorname{rot}^2 = \operatorname{rot} \operatorname{rot}$. Therefore, formulation of the operator rot^2 in $(1.9)_1$ is right.

To satisfy boundary conditions $(1.9)_3$ we have to introduce the tangent and normal vectors to S_1 and S_2 . From the geometry of a cylinder we have $\bar{n}|_{S_1} = \bar{n}|_{S_0}$. Let S_0 be described by a sufficiently regular function $\varphi(x_1, x_2) = 0$. Then $\bar{n}|_{S_0} = \frac{\nabla \varphi}{|\nabla \varphi|}$. Next the tangent vector to S_0 , denoted by $\bar{\tau}$, equals $\bar{\tau} = \frac{\bar{\nabla} \varphi}{|\bar{\nabla} \varphi|}$, where $\bar{\nabla} \varphi = (-\varphi_{,x_2}, \varphi_{,x_1})$. We have two tangent vectors to S_1 : $\bar{\tau}_1 = (\bar{\tau}, 0)$, $\bar{\tau}_2 = (0, 0, 1)$. On S_2 we have $\bar{n}|_{S_2} = (0, 0, 1)$, $\bar{\tau}_1|_{S_2} = (1, 0, 0)$, $\bar{\tau}_2|_{S_2} = (0, 1, 0)$.

Since $\bar{n}|_{S_1} = \bar{n}|_{S_0}$ we have that $u \cdot \bar{n} = 0$ on S_1 . We have also that $w \cdot \bar{n}|_{S_2} = 0$, so finally $u \cdot \bar{n}|_S = 0$ holds. Next we examine the condition

$$(1.10) \quad \bar{n} \times \operatorname{rot} w|_S = 0$$

On S_1 it is equivalent to

$$(1.11) \quad \begin{aligned} \bar{\tau}_1 \cdot \operatorname{rot} w|_{S_1} &= 0, \\ \bar{\tau}_2 \cdot \operatorname{rot} w|_{S_1} &= 0, \end{aligned}$$

where the second condition $(1.11)_2$ equals $w_{1,x_2} - w_{2,x_1}|_{S_1} = 0$ so $\text{rot}^{(2)}w|_{S_1} = 0$ in view of boundary condition $(1.5)_4$. To satisfy $(1.11)_1$ we express it explicitly in the form

$$\tau_1(w_{2,x_3} - w_{3,x_2}) + \tau_2(w_{3,x_1} - w_{1,x_3})|_{S_1} = 0.$$

It holds because $w_{i,x_3} = 0$, $i = 1, 2$, and $w_3 = 0$.

Similarly, we show that $\bar{n} \times \text{rot}w|_{S_2} = 0$. Hence (1.10) holds.

Now we present results of this paper. The introduced norms in these formulations are defined in Section 2. Remark 3.3 yields

Theorem 1. *Let $T > 0$ be given. Let $w(0) \in H^1(\Omega)$, $\bar{h} \in L_2(kT, (k+1)T; L_2(\Omega))$ for any $k \in \mathbb{N}_0$. Then there exists a solution to problem (1.7) such that $w \in V_2^1(kT, (k+1)T; \Omega)$ and*

$$\|w(kT)\|_{H^1(\Omega)} \leq c\bar{A}_1,$$

$$\|w(t)\|_{H^1(\Omega)}^2 + \int_{kT}^t \|w(t')\|_{H^2(\Omega)}^2 dt' \leq c\bar{A}_1^2,$$

where $t \in (kT, (k+1)T)$, $k \in \mathbb{N}_0$ and

$$\bar{A}_1^2 = \sup_k \frac{1}{\nu} \int_{kT}^{(k+1)T} \|\bar{h}(t)\|_{L_2(\Omega)}^2 dt + \|w(0)\|_{H^1(\Omega)}^2.$$

Lemma 3.4 implies

Theorem 2. *Let the assumptions of Theorem 1 hold. Let $w(0) \in B_{\sigma,2}^1(\Omega)$, $\bar{h} \in L_2(kT, (k+1)T; L_\sigma(\Omega))$, $\sigma > 3$, $k \in \mathbb{N}_0$. Then*

$$\|w(t)\|_{W_\sigma^1(\Omega)} + \|w\|_{W_{\sigma,2}^{2,1}(\Omega \times (kT, (k+1)T))} \leq \bar{A}_2,$$

where \bar{A}_2 depends on \bar{A}_1 ,

$$\|w(0)\|_{B_{\sigma,2}^1(\Omega)}, \left(\sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\bar{h}(t)\|_{L_\sigma(\Omega)}^2 dt \right)^{1/2}.$$

From Lemma 4.2 we have

Theorem 3. *Let the assumptions of Theorems 1 and 2 hold. Let $c_* \in (0, \nu]$. Let γ_* be so small that $\frac{c_*}{2} \leq \nu - \frac{c_0}{\nu^3} \gamma_*^4$, where c_0 is the constant from (4.20). Let $\gamma \leq \gamma_*$. Let*

$$\begin{aligned} \|u(0)\|_{H^1(\Omega)}^2 &\leq \gamma, \\ \frac{c}{\nu} \|g(t)\|_{L_2(D)}^2 &\leq \frac{c_*}{4} \gamma. \end{aligned}$$

Then

$$\|u(t)\|_{H^1(\Omega)}^2 \leq \gamma \quad \text{for } t \in \mathbb{R}_+.$$

From Theorems 1 and 3 and Remarks 3.5 and 4.3 we have

Theorem 4. *Assume that $f \in L_2(kT, (k+1)T; L_2(\Omega))$, $v(0) \in H^1(\Omega)$, $k \in \mathbb{N}_0$. Then there exists a global regular solution to (1.1) such that $v = w + u$, $p = \eta + q$, $f = h + g$ and $v \in W_2^{2,1}(\Omega \times (kT, (k+1)T))$, $\nabla p \in L_2(\Omega \times (kT, (k+1)T))$.*

In [22] problem (1.1) is considered in the periodic box. In this case global existence of two-dimensional solutions such that $w \in V_2^1(kT, (k+1)T; \Omega)$ is proved under very restrictive relation between T , ν , \bar{A}_1 . The relation holds for sufficiently large T , ν and correspondingly small \bar{A}_1 . In this paper we omit the restriction by considering problem (1.1) in a cylindrical domain with the Navier boundary condition. This, in view of (2.16), gives that the estimate

$$(1.12) \quad \|w\|_{W_2^1(kT, (k+1)T; \Omega)} \leq c_0, \quad k \in \mathbb{N}_0,$$

holds without any restrictions. The main aim in [22] and in this paper is to show that constant c_0 in (1.12) does not depend on k .

This guarantees that the two-dimensional solutions does not increase in time. Hence, also stability of two-dimensional solutions can be proved.

The first results connected with the stability of global regular solutions to the nonstationary Navier-Stokes equations were proved by Beirao da Veiga and Secchi [3], followed by Ponce, Racke, Sideris and Titi [18]. Paper [3] is concerned with the stability in L_p -norm of a strong three-dimensional solution of the Navier-Stokes system with zero external force in the whole space. In [18], assuming that the external force is zero and a three-dimensional initial function is close to a two-dimensional one in $H^1(\mathbb{R}^3)$, the authors showed the existence of a global strong solution in \mathbb{R}^3 which remains close to a two-dimensional strong solution for all times. In [17] Mucha obtained a similar result under weaker assumptions about the smallness of the initial velocity perturbation.

In the class of weak Leray-Hopf solutions the first stability result was obtained by Gallagher [8]. She proved the stability of two-dimensional solutions of the Navier-Stokes equations with periodic boundary conditions under three-dimensional perturbations both in L_2 and $H^{\frac{1}{2}}$ norms.

The stability of nontrivial periodic regular solutions to the Navier-Stokes equations was studied by Iftimie [10] and by Mucha [15]. The paper [15] is devoted to the case when the external force is a potential belonging to $L_{r,loc}(\mathbb{T}^3 \times [0, \infty))$ and when the initial data belongs to the space $W_r^{2-2/r}(\mathbb{T}^3) \cap L_2(\mathbb{T}^3)$, where $r \geq 2$ and \mathbb{T} is a torus. Under the assumption that there exists a global solution with data of regularity mentioned above and assuming that small perturbations of data have the same regularity as above, the author proves that perturbations of the velocity and the gradient of the pressure remain small in the spaces $W_r^{2,1}(\mathbb{T}^3 \times (k, k+1))$ and $L_r(\mathbb{T}^3 \times (k, k+1))$, $k \in \mathbb{N}$, respectively. Paper [10] contains results concerning the stability of two-dimensional regular solutions to the Navier-Stokes system in a three-dimensional torus but here the initial data in the three-dimensional problem belong to an anisotropic space of functions having different regularity in the first two directions than in the third direction, and the external force vanishes. Moreover, Mucha [16] studies the stability of regular solutions to the nonstationary Navier-Stokes system in \mathbb{R}^3 assuming that they tend in $W_r^{2,1}$ spaces ($r \geq 2$) to constant flows.

The papers of Auscher, Dubois and Tchamitchian [1] and of Gallagher, Iftimie and Planchon [9] concern the stability of global regular solutions to the Navier-Stokes equations in the whole space \mathbb{R}^3 with zero external force. These authors show that the norms of the considered solutions decay as $t \rightarrow \infty$.

It is worth mentioning the paper of Zhou [25], who proved the asymptotic stability of weak solutions u with the property: $u \in L_2(0, \infty, BMO)$ to the Navier-Stokes equations in \mathbb{R}^n , $n \geq 3$, with a force vanishing as $t \rightarrow \infty$.

An interesting result was obtained by Karch and Pilarczyk [11], who concentrate on the stability of Landau solutions to the Navier-Stokes system in \mathbb{R}^3 . Assuming that the external force is a singular distribution they prove the asymptotic stability of the solution under any L_2 -perturbation.

Paper [7] of Chemin and Gallagher is devoted to the stability of some unique global solution with large data in a very weak sense.

Finally, the stability of Leray-Hopf weak solutions has recently been examined by Bardos et al. [2], where equations with vanishing external force are considered. That paper concerns the following three cases: two-dimensional flows in infinite cylinders under three-dimensional perturbations which are periodic in the vertical direction; helical flows in circular

cylinders under general three-dimensional perturbations; and axisymmetric flows under general three-dimensional perturbations. The theorem concerning the first case extends a result obtained by Gallagher [8] for purely periodic boundary conditions.

Most of the papers discussed above concern to the case with zero external force ([1–3], [7–10], [17], [18]) or with force which decays as $t \rightarrow \infty$ ([18]). Exceptions are [11, 15, 16], where very special external forces, which are singular distributions in [11] or potentials in [15–16], are considered. However, the case of potential forces is easily reduced to the case of zero external forces.

The aim of our paper is to prove the stability result for a large class of external forces f_s which do not produce solutions decaying as $t \rightarrow \infty$.

It is essential that our stability results are obtained together with the existence of a global strong three-dimensional solution close to a two-dimensional one.

The paper is divided into two main parts. In the first we prove existence of global strong two-dimensional solutions not vanishing as $t \rightarrow \infty$ because the external force does not vanish either. To prove existence of such solutions we use the step by step method. For this purpose we have to show that the data in the time interval $[kT, (k+1)T]$, $k \in \mathbb{N}$, do not increase with k . We do not need any restrictions on the time step T . In the second part we prove existence of three-dimensional solutions that remain close to two-dimensional solutions. For this we need the initial velocity and the external force to be sufficiently close in appropriate norms to the initial velocity and the external force of the two-dimensional problems.

The proofs of this paper are based on the energy method. Thanks to the Navier boundary conditions the nonlinear term in the two-dimensional Navier-Stokes equations does not have any influence on the form of the energy estimate. The proofs of global existence which follow from the step by step technique are possible thanks to the natural decay property of the Navier-Stokes equations. This is mainly used in the first part of the paper (Section 3). To prove stability (Section 4) we use smallness of data $(v(0) - v_s(0)), (f - f_s)$ and a contradiction argument applied to the nonlinear ordinary differential inequality (4.24). The paper is a generalization of results from [22, 24], where the periodic case is considered.

We restrict ourselves to prove estimates only, because existence follows easily by the Faedo-Galerkin method.

The paper is organized as follows. In Section 2 we introduce notation and give some auxiliary results. Section 3 is devoted to the existence of a two-dimensional solution. It also contains some useful estimates of the

solution. In Section 4 we prove the existence of a global strong solution to problem (1.1) close to the two-dimensional solution for all time.

2. Notation and auxiliary results

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $L_p(\Omega)$, $p \in [1, \infty]$, $\Omega \subset \mathbb{R}^n$ we denote the Lebesgue space of integrable functions and by $H^s(\Omega)$, $s \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^n$, the Sobolev space of function with the finite norm

$$\|u\|_{H^s} = \|u\|_{H^s(\Omega)} = \left(\sum_{|\alpha| \leq s} \int_{\Omega} |D_x^\alpha u(x)|^2 dx \right)^{1/2},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha_i \in \mathbb{N}_0$, $i = 1, \dots, n$, $n = 2, 3$. Let $u = (u_1, \dots, u_n)$ be a vector. Then $|u| = \sqrt{u_1^2 + \dots + u_n^2}$.

Next we introduce the anisotropic Lebesgue and Sobolev spaces with the mixed norms. $L_{p_1, p_2}(\Omega \times (0, T))$ and $W_{p_1, p_2}^{s, s/2}(\Omega \times (0, T))$, $p_1, p_2 \in (1, \infty)$, $\frac{s}{2} \in \mathbb{N}$, are spaces with the following finite norms

$$\|u\|_{L_{p_2}(0, T; L_{p_1}(\Omega))} \equiv \|u\|_{L_{p_1, p_2}(\Omega^T)} = \left(\int_0^T \left(\int_{\Omega} |u(x, t)|^{p_1} dx \right)^{p_2/p_1} dt \right)^{1/p_1} < \infty,$$

$$\|u\|_{W_{p_1, p_2}^{s, s/2}(\Omega^T)} = \sum_{|\alpha| + 2\alpha_0 \leq s} \left(\int_0^T \left(\int_{\Omega} |D_x^\alpha \partial_t^{\alpha_0} u(x, t)|^{p_1} dx \right)^{p_2/p_1} dt \right)^{1/p_1} < \infty,$$

where s is even.

From [4, Ch. 4, Sect. 18] we recall the definition of the Besov spaces used in this paper. By $B_{p, q}^l(\Omega)$, $l \in \mathbb{R}_+$, $p, q \in [1, \infty]$, $\Omega \subset \mathbb{R}^n$, we denote the linear normed space of functions $u = u(x)$, $x \in \Omega$, with the finite norm

$$\|u\|_{B_{p, q}^l(\Omega)} = \|u\|_{L_p(\Omega)} + \sum_{i=1}^n \left(\int_0^{h_0} \left| \frac{\Delta_i^m(h; \Omega) \partial_i^k u}{h^{l-k}} \right|_{L_p(\Omega)}^q \frac{dh}{h} \right)^{1/q},$$

where $m > l - k > 0$, $m, k \in \mathbb{N}$, $\Delta_i(h)u(x) = u(x + h\bar{e}_i) - u(x)$, $\Delta_i^m(h)u(x) = \Delta_i(\Delta_i^{m-1}(h)u(x))$, \bar{e}_i – versor of the i -th axis, $i = 1, \dots, n$. Moreover,

$$\Delta_i^m(h; \Omega)u(x) = \begin{cases} \Delta_i^m(h; \Omega)u(x) & \text{for } [x, x + mhe_i] \in \Omega, \\ 0 & \text{for } [x, x + mhe_i] \notin \Omega. \end{cases}$$

In this paper the energy method is used to show main estimates and existence. For this purpose we need space $V_2^k(T_1, T_2; \Omega)$, $k \in \mathbb{N}$, with the finite norm

$$\|u\|_{V_2^k(T_1, T_2; \Omega)} = \left(\text{esssup}_{T_1 \leq t \leq T_2} \|u(t)\|_{H^k(\Omega)}^2 + \int_{T_1}^{T_2} \|u(t)\|_{H^{k+1}(\Omega)}^2 dt \right)^{1/2}.$$

Let us consider the problem

$$(2.1) \quad \begin{aligned} w_{2,x_1} - w_{1,x_2} &= b & \text{in } \Omega, \\ w_{1,x_1} + w_{2,x_2} &= 0 & \text{in } \Omega, \\ w \cdot \bar{n} &= 0 & \text{on } S_0, \end{aligned}$$

Lemma 2.1. *Assume that $b \in H^s(\Omega)$, $s \in \mathbb{N}_0$. Then there exists a solution to problem (2.1) such that $w \in H^{s+1}(\Omega)$ and*

$$(2.2) \quad \|w\|_{H^{s+1}(\Omega)} \leq c \|b\|_{H^s(\Omega)}.$$

Proof. Equation (2.1)₂ implies existence of potential ψ such that $w_1 = -\psi_{,x_2}$, $w_2 = \psi_{,x_1}$. Then problem (2.1) takes the form

$$(2.3) \quad \begin{aligned} \Delta^{(2)} \psi &= b & \text{in } \Omega, \\ \bar{\tau} \cdot \nabla \psi &= 0 & \text{on } S_0. \end{aligned}$$

The boundary condition (2.3)₂ implies that $\psi = \text{const}$ on S_0 . Hence, in view of the definition of ψ , we can assume that $\psi = 0$ on S_0 . Therefore (2.3) implies the Dirichlet problem

$$(2.4) \quad \begin{aligned} \Delta^{(2)} \psi &= b & \text{in } \Omega, \\ \psi &= 0 & \text{on } S_0. \end{aligned}$$

Problem (2.4) yields existence of ψ in $H^{s+2}(\Omega)$ and the estimate

$$\|\psi\|_{H^{s+2}(\Omega)} \leq c \|b\|_{H^s(\Omega)},$$

so (2.2) holds. This concludes the proof. \square

Lemma 2.2. *Let $\nabla u \in L_2(\Omega)$ and $u|_{S_0} = 0$. Then the following Poincaré inequality holds*

$$(2.5) \quad c_p \|u\|_{L_2(\Omega)} \leq \|\nabla u\|_{L_2(\Omega)}.$$

Let us consider the elliptic overdetermined problem

$$(2.6) \quad \begin{aligned} \operatorname{rot} u &= b & \text{in } D, \\ \operatorname{div} u &= 0 & \text{in } D, \\ u \cdot \bar{n} &= 0 & \text{on } S, \end{aligned}$$

where D is a cylinder and the following compatibility condition holds

$$(2.7) \quad \operatorname{div} b = 0.$$

Lemma 2.3. (see also [21]) *Let $b \in H^i(D)$, $i = 1, 2$, and satisfy (2.7). Then there exists a solution to (2.6) such that $u \in H^{i+1}(D)$ and*

$$(2.8) \quad \|u\|_{H^{i+1}(D)} \leq c_e \|b\|_{H^i(D)}, \quad i = 0, 1, \quad H^0(D) = L_2(D),$$

where c_e does not depend on u .

Proof. By Lemma 1 from [6], (2.6)_{2,3} imply existence of a vector e such that

$$(2.9) \quad u = \operatorname{rote}, \quad \operatorname{div} e = 0, \quad e_\tau|_S = 0.$$

The explicit construction of such vector is presented in [6] and also in [23, Sect. 3]. In view of (2.9) problem (2.6) takes the form

$$(2.10) \quad -\Delta e = b, \quad e_\tau|_S = 0, \quad \operatorname{div} e|_S = 0,$$

where $e_\tau = e \cdot \bar{\tau}$. The second boundary condition in (2.10) guarantees that $\operatorname{div} e = 0$ in D .

Recalling the normal and tangent vectors to S ,

$$\begin{aligned} \bar{n}|_{S_1} &= \frac{\nabla \varphi}{|\nabla \varphi|}, \quad \bar{\tau}_1|_{S_1} = \frac{\nabla^\perp \varphi}{|\nabla \varphi|}, \quad \bar{\tau}_2|_{S_1} = (0, 0, 1), \\ \bar{n}|_{S_2} &= (0, 0, 1), \quad \bar{\tau}_1|_{S_2} = (1, 0, 0), \quad \bar{\tau}_2|_{S_2} = (0, 1, 0), \end{aligned}$$

we express problem (2.10) in the form

$$(2.11) \quad \begin{aligned} -\Delta e &= b & \text{in } D, \\ e_{\tau_1} &= 0, \quad e_{\tau_2} = 0, \quad \bar{n} \cdot \nabla e_n + e_n \operatorname{div} \bar{n} = 0 & \text{on } S_1, \\ e_1 &= 0, \quad e_2 = 0, \quad \frac{\partial}{\partial x_3} e_3 = 0 & \text{on } S_2, \end{aligned}$$

where $e_n = e \cdot \bar{n}$. To obtain the last boundary conditions on S_1 and S_2 we formulate dive in the curvilinear coordinates corresponding to vectors \bar{n} , $\bar{\tau}_1$, $\bar{\tau}_2$ and project it on S .

To prove existence of solutions to problem (2.11) we use the idea of regularizer (see [14, Ch. 4]). For this we need a partition of unity and appropriate local estimates. To get (2.8) we need the local estimates in H^2 . Such estimates are easily proved in neighborhoods of interior points and points on the smooth part of S , so points located in a positive distance from the edge $\bar{S}_1 \cap \bar{S}_2$.

Since domain D contains right angles between S_1 and S_2 we are not able to obtain the needed estimates in neighborhoods of points of the edge. From (2.11)₃ it follows that on S_2 we have the Dirichlet and the Neumann conditions for the Poisson equation. Therefore, we can reflect the solutions of the problems with respect to S_2 . Then the necessary estimates can be easily derived. We have to mention that local estimates near S_1 are shown after its local flattening.

Summarizing, we have existence of solutions to (2.11) such that $e \in H^2(D)$ and

$$\|e\|_{H^2(D)} \leq c\|b\|_{L_2(D)}.$$

This implies (2.8) for $i = 0$. Similarly, we have (2.8) for $i = 1$. This concludes the proof. \square

Let u satisfy

$$(2.12) \quad \begin{aligned} \operatorname{div} u &= 0 \quad \text{in } D, \\ u \cdot \bar{\tau} &= 0 \quad \text{on } S. \end{aligned}$$

Lemma 2.4. (see [6, Lemma 2.1]) *For any u satisfying (2.12) the inequality holds*

$$(2.13) \quad \|u\|_{H^{i+1}(D)} \leq c\|\operatorname{rot} u\|_{H^i(D)}, \quad i = 0, 1,$$

where c does not depend on u .

We also need the direct and inverse trace theorems for spaces with mixed norms.

Lemma 2.5. (see [5])

- (i) Let $u \in W_{p,p_0}^{s,s/2}(\Omega^T)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $p, p_0 \in (1, \infty)$. Then $u(x, t_0) = u(x, t)|_{t=t_0}$ for $t_0 \in [0, T]$ belongs to $B_{p,p_0}^{s-2/p_0}(\Omega)$ and

$$(2.14) \quad \|u(\cdot, t_0)\|_{B_{p,p_0}^{s-2/p_0}(\Omega)} \leq c\|u\|_{W_{p,p_0}^{s,s/2}(\Omega^T)},$$

where c does not depend on u .

- (ii) For a given $\tilde{u} \in B_{p,p_0}^{s-2/p_0}(\Omega)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $(p, p_0) \in (1, \infty)$, there exists a function $u \in W_{p,p_0}^{s,s/2}(\Omega^T)$, such that $u|_{t=t_0} = \tilde{u}$ for $t_0 \in [0, T]$ and

$$(2.15) \quad \|u\|_{W_{p,p_0}^{s,s/2}(\Omega^T)} \leq c \|\tilde{u}\|_{B_{p,p_0}^{s-2/p_0}(\Omega)},$$

where c does not depend on u .

Lemma 2.6. For sufficiently regular solutions to (1.7) the following formula is valid

$$(2.16) \quad \int_{\Omega} w \cdot \nabla w \cdot \Delta w dx = - \int_{S_0} (w \cdot \nabla w_1 n_2 - w \cdot \nabla w_2 n_1) \text{rot}^{(2)} w dS_0,$$

where n_1, n_2 are the Cartesian coordinates of the normal vector \bar{n} to S_0 and the r.h.s. of (2.16) vanishes by (1.7)₃.

Proof. Let $\text{rot}^{(2)} w = w_{2,x_1} - w_{1,x_2}$ and $\tilde{\text{rot}} \varphi = (\varphi_{,x_2} - \varphi_{,x_1})$. Then

$$\begin{aligned} \tilde{\text{rot}} \text{rot}^{(2)} w &= \begin{pmatrix} (\text{rot}^{(2)} w)_{,x_2} \\ -(\text{rot}^{(2)} w)_{,x_1} \end{pmatrix} = \begin{pmatrix} (w_{2,x_1} - w_{1,x_2})_{,x_2} \\ -(w_{2,x_1} - w_{1,x_2})_{,x_1} \end{pmatrix} \\ &= \begin{pmatrix} w_{2,x_1 x_2} - w_{1,x_2 x_2} \\ w_{1,x_2 x_1} - w_{2,x_1 x_1} \end{pmatrix} \equiv I. \end{aligned}$$

Using the continuity equation (1.7)₂ we have

$$w_{2,x_1 x_2} = -w_{1,x_1 x_1}, \quad w_{1,x_1 x_2} = -w_{2,x_2 x_2}.$$

Then $I = -\Delta w$, so

$$(2.17) \quad \tilde{\text{rot}} \text{rot}^{(2)} w = -\Delta w.$$

Using (2.17) we have

$$\begin{aligned} (2.18) \quad & \int_{\Omega} w \cdot \nabla w \cdot \Delta w dx = - \int_{\Omega} w \cdot \nabla w \tilde{\text{rot}} \text{rot}^{(2)} w dx \\ &= - \int_{\Omega} [w \cdot \nabla w_1 (\text{rot}^{(2)} w)_{,x_2} - w \cdot \nabla w_2 (\text{rot}^{(2)} w)_{,x_1}] dx \\ &= \int_{\Omega} [(w \cdot \nabla w_2 \text{rot}^{(2)} w)_{,x_1} - (w \cdot \nabla w_1 \text{rot}^{(2)} w)_{,x_2}] dx \\ &\quad - \int_{\Omega} [(w \cdot \nabla w_2)_{,x_1} \text{rot}^{(2)} w - (w \cdot \nabla w_1)_{,x_2} \text{rot}^{(2)} w] dx \equiv I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{S_0} (w \cdot \nabla w_2 n_1 - w \cdot \nabla w_1 n_2) \text{rot}^{(2)} w dS_0$$

and

$$\begin{aligned} I_2 &= - \int_{\Omega} (w \cdot \nabla w_{2,x_1} - w \cdot \nabla w_{1,x_2}) \text{rot}^{(2)} w dx \\ &\quad - \int_{\Omega} [(w_{1,x_1} w_{2,x_1} + w_{2,x_1} w_{2,x_2} - w_{1,x_2} w_{1,x_1} - w_{2,x_2} w_{1,x_2}) \text{rot}^{(2)} w] dx \\ &\equiv I_2^1 + I_2^2. \end{aligned}$$

Continuing,

$$\begin{aligned} I_2^1 &= - \int_{\Omega} w \cdot \nabla \text{rot}^{(2)} w \text{rot}^{(2)} w dx = - \frac{1}{2} \int_{\Omega} w \cdot \nabla (\text{rot}^{(2)} w)^2 dx \\ &= \frac{1}{2} \int_{S_0} w \cdot \bar{n} (\text{rot}^{(2)} w)^2 dS_0 = 0 \end{aligned}$$

and

$$I_2^2 = - \int_{\Omega} \text{div } w (\text{rot}^{(2)} w)^2 dx = 0.$$

Hence (2.16) holds. This concludes the proof. \square

Since sometimes is more appropriate to use the slip boundary condtions we find a relation between the Navier and the slip boundary conditions.

Lemma 2.7. *Let $\mathbb{D}^{(2)}(w) = \{w_{i,x_j} + w_{j,x_i}\}_{i,j=1,2}$, $\text{rot}^{(2)} w = w_{2,x_1} - w_{1,x_2}$. Let $\bar{n} = (n_1, n_2)$ be the normal unit outward vector to S_0 and $\bar{\tau} = (-n_2, n_1)$ be the tangent vector. Then*

$$(2.19) \quad \bar{n} \cdot \mathbb{D}^{(2)}(w) \cdot \bar{\tau} = \text{rot}^{(2)} w + 2w_{n,\tau} - 2w_i n_{i,\tau},$$

where $w_n = w \cdot \bar{n}$, $w_{,\tau} = \bar{\tau} \cdot \nabla w$.

Proof.

$$\begin{aligned} \bar{n} \cdot \mathbb{D}^{(2)}(w) \cdot \bar{\tau} &= n_i (w_{i,x_j} + w_{j,x_i}) \tau_j = n_i (-w_{i,x_j} + w_{j,x_i}) \tau_j \\ &\quad + 2n_i w_{i,x_j} \tau_j = n_1 (-w_{1,x_2} + w_{2,x_1}) \tau_2 + n_2 (-w_{2,x_1} + w_{1,x_2}) \tau_1 \\ &\quad + 2n_i w_{i,x_j} \tau_j = (n_1 \tau_2 - n_2 \tau_1) \text{rot}^{(2)} w + 2w_{n,\tau} - 2w_i n_{i,\tau} \\ &= \text{rot}^{(2)} w + 2w_{n,\tau} - 2w_i n_{i,\tau}. \end{aligned}$$

This concludes the proof.

Let us consider the Stokes problem

$$\begin{aligned}
(2.20) \quad & w_t + \nu \tilde{\text{rot}} \text{rot}^{(2)} w + \nabla \eta = f \quad \text{in } \Omega \times (0, T), \\
& \text{div } w = 0 \quad \text{in } \Omega \times (0, T), \\
& w \cdot \bar{n} = 0, \quad \text{rot}^{(2)} w = 0 \quad \text{on } S_0 \times (0, T), \\
& w|_{t=0} = w(0) \quad \text{in } \Omega.
\end{aligned}$$

The theory developed in [12, 13, 19, 20] implies

Lemma 2.8. *Let $f \in L_{\sigma_1, \sigma_2}(\Omega \times (0, T))$, $w(0) \in B_{\sigma_1, \sigma_2}^{2-2/\sigma_2}(\Omega)$, $\sigma_1, \sigma_2 \in (1, \infty)$. Then there exists a solution to problem (2.20) such that $w \in W_{\sigma_1, \sigma_2}^{2,1}(\Omega \times (0, T))$, $\nabla \eta \in L_{\sigma_1, \sigma_2}(\Omega \times (0, T))$ and*

$$\begin{aligned}
(2.21) \quad & \|w\|_{W_{\sigma_1, \sigma_2}^{2,1}(\Omega \times (0, T))} + \|\nabla \eta\|_{L_{\sigma_1, \sigma_2}(\Omega \times (0, T))} \\
& \leq c(\|f\|_{L_{\sigma_1, \sigma_2}(\Omega \times (0, T))} + \|w(0)\|_{B_{\sigma_1, \sigma_2}^{2-2/\sigma_2}(\Omega)}),
\end{aligned}$$

where c may depend on T .

3. Two-dimensional solutions

First we have

Lemma 3.1. *Assume that $\bar{h} \in L_{2, \text{loc}}(\mathbb{R}_+; L_2(\Omega))$, $w(0) \in L_2(\Omega)$. Assume that $T > 0$ is given. Denote $A_1^2 = \sup_{k \in \mathbb{N}_0} \frac{1}{\nu c_1} \int_{kT}^{(k+1)T} \|\bar{h}(t)\|_{L_2(\Omega)}^2 dt < \infty$, $A_2^2 = \frac{A_1^2}{1 - e^{-\nu c_1 T}} + \|w(0)\|_{L_2(\Omega)}^2 < \infty$, where c_1 appearing in (3.5) follows from Lemma 2.1 (see (2.2)). Then for solutions to (1.7) we have*

$$(3.1) \quad \|w(kT)\|_{L_2(\Omega)}^2 \leq A_2^2$$

and

$$(3.2) \quad \|w(t)\|_{L_2(\Omega)}^2 + \nu c_1 \int_{kT}^t \|w(t')\|_{H^1(\Omega)}^2 dt' \leq A_1^2 + A_2^2 \equiv A_3^2,$$

where $t \in (kT, (k+1)T]$ and $k \in \mathbb{N}_0$.

Proof. Multiplying (1.7)₁ by w and integrating over Ω yields

$$(3.3) \quad \frac{1}{2} \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 + \nu \int_{\Omega} \tilde{\text{rot}} \text{rot}^{(2)} w \cdot w dx = \int_{\Omega} \bar{h} \cdot w dx,$$

where the first of boundary conditions $(1.7)_3$ is used.
The second term on the l.h.s. of (3.3) equals

$$(3.4) \quad \begin{aligned} & \int_{\Omega} [(\operatorname{rot}^{(2)} w)_{,x_2} w_1 - (\operatorname{rot}^{(2)} w)_{,x_1} w_2] dx \\ &= \int_{\Omega} [(\operatorname{rot}^{(2)} w w_1)_{,x_2} + (-\operatorname{rot}^{(2)} w w_2)_{,x_1}] dx + \int_{\Omega} |\operatorname{rot}^{(2)} w|^2 dx. \end{aligned}$$

Applying the Green formula, the first term on the r.h.s. of (3.4) is equal to

$$\int_{S_0} \operatorname{rot}^{(2)} w (w_1 n_2 - w_2 n_1) dS_0 = 0,$$

where the second condition from $(1.7)_3$ is utilized.

Employing (3.4), Lemma 2.1 and the Hölder and the Young inequalities to the r.h.s. of (3.3) we obtain from (3.3) the inequality

$$(3.5) \quad \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 + \nu c_1 \|w\|_{H^1(\Omega)}^2 \leq \frac{1}{\nu c_1} \|\bar{h}\|_{L_2(\Omega)}^2.$$

Expressing (3.5) in the form

$$(3.6) \quad \frac{d}{dt} \|w\|_{L_2(\Omega)}^2 + \nu c_1 \|w\|_{L_2(\Omega)}^2 \leq \frac{1}{\nu c_1} \|\bar{h}\|_{L_2(\Omega)}^2$$

we integrate it with respect to time from $t = kT$ to $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$, to derive

$$(3.7) \quad \begin{aligned} \|w(t)\|_{L_2(\Omega)}^2 &\leq \frac{1}{\nu c_1} \int_{kT}^t \|\bar{h}(t')\|_{L_2(\Omega)}^2 dt' \\ &\quad + \|w(kT)\|_{L_2(\Omega)}^2 \exp(-\nu c_1(t - kT)). \end{aligned}$$

Setting $t = (k+1)T$ inequality (3.7) implies

$$(3.8) \quad \|w((k+1)T)\|_{L_2(\Omega)}^2 \leq \frac{1}{\nu c_1} \int_{kT}^t \|\bar{h}(t)\|_{L_2(\Omega)}^2 dt + \|w(kT)\|_{L_2(\Omega)}^2 e^{-\nu c_1 T}.$$

By iteration we get

$$(3.9) \quad \|w(kT)\|_{L_2(\Omega)}^2 \leq \frac{A_1^2}{1 - e^{-\nu c_1 T}} + \|w(0)\|_{L_2(\Omega)}^2 e^{-\nu c_1 kT} \leq A_2^2,$$

so (3.1) holds. Integrating (3.5) with respect to time from $t = kT$ to $t \in (kT, (k+1)T]$ and using (3.1) yields (3.2). This concludes the proof. \square

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Let $w(0) \in H^1(\Omega)$.*

Let $A_4^2 = \frac{c_1 A_1^2}{1 - e^{-c_p \nu T}} + \|\text{rot}^{(2)}(0)\|_{L_2(\Omega)}^2$. Then

$$(3.10) \quad \|\text{rot}^{(2)}w(kT)\|_{L_2(\Omega)}^2 \leq A_4^2$$

and

$$(3.11) \quad \begin{aligned} & \|\text{rot}^{(2)}w(t)\|_{L_2(\Omega)}^2 + \nu c_p \int_{kT}^t \|\text{rot}^{(2)}w(t')\|_{H^1(\Omega)}^2 dt' \\ & \leq \frac{1}{\nu} \int_{kT}^t \|\bar{h}(t')\|_{L_2(\Omega)}^2 dt' + \|\text{rot}^{(2)}w(kT)\|_{L_2(\Omega)}^2 \leq c_1 A_1^2 + A_4^2 \equiv A_5^2. \end{aligned}$$

Proof. Multiplying (1.7)₁ by $-\Delta w$ and integrating the result over Ω yields

$$(3.12) \quad \begin{aligned} & - \int_{\Omega} w_{,t} \cdot \Delta w dx + \nu \int_{\Omega} |\Delta w|^2 dx = \int_{\Omega} \nabla \bar{\eta} \cdot \Delta w dx + \int_{\Omega} w \cdot \nabla w \cdot \Delta w dx \\ & - \int_{\Omega} \bar{h} \cdot \Delta w dx. \end{aligned}$$

Using that $\Delta w = -\tilde{\text{rot}}\text{rot}^{(2)}w$ the first term on the l.h.s. of (3.12) equals

$$\begin{aligned} & \int_{\Omega} w_t \tilde{\text{rot}}\text{rot}^{(2)}w dx = \int_{\Omega} [w_{1,t} \partial_{x_2} \text{rot}^{(2)}w - w_{2,t} \partial_{x_1} \text{rot}^{(2)}w] dx \\ & = \int_{\Omega} [(-w_{2,t} \text{rot}^{(2)}w)_{,x_1} + (w_{1,t} \text{rot}^{(2)}w)_{,x_2}] dx \\ & + \int_{\Omega} [w_{2,x_1} \text{rot}^{(2)}w - w_{1,x_2} \text{rot}^{(2)}w] dx = \int_{\Omega} \text{rot}^{(2)}w \partial_t \text{rot}^{(2)}w dx \\ & = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\text{rot}^{(2)}w|^2 dx, \end{aligned}$$

where we used the boundary conditions (1.7)₃. In view of the boundary conditions (1.7)₃ also and Lemma 2.6 the second term on the r.h.s. of (3.12) vanishes. The first term on the r.h.s. of (3.12) takes the form

$$\begin{aligned} & - \int_{\Omega} [\partial_{x_1} \bar{\eta} (\text{rot}^{(2)}w)_{,x_2} - \partial_{x_2} \bar{\eta} (\text{rot}^{(2)}w)_{,x_1}] dx \\ & = \int_{\Omega} [\partial_{x_2} (\partial_{x_1} \bar{\eta} \text{rot}^{(2)}w) + \partial_{x_1} (-\partial_{x_2} \bar{\eta} \text{rot}^{(2)}w)] dx \end{aligned}$$

which also vanishes in view of boundary conditions (1.7)₃.

Using the above calculations in (3.12) yields

$$(3.13) \quad \frac{1}{2} \frac{d}{dt} \|\text{rot}^{(2)} w\|_{L_2(\Omega)}^2 + \nu \|\nabla \text{rot}^{(2)} w\|_{L_2(\Omega)}^2 = \int_{\Omega} \bar{h} \cdot \Delta w dx$$

Applying the Hölder and the Young inequalities to the r.h.s. of (3.13) and using that $|\Delta w| = |\nabla \text{rot}^{(2)} w|$ we obtain

$$(3.14) \quad \frac{d}{dt} \|\text{rot}^{(2)} w\|_{L_2(\Omega)}^2 + \nu \|\nabla \text{rot}^{(2)} w\|_{L_2(\Omega)}^2 \leq \frac{1}{\nu} \|\bar{h}\|_{L_2(\Omega)}^2$$

Since $\text{rot}^{(2)} w|_{S_0} = 0$ we can apply the Poincaré inequality (see (2.5)) to (3.14). Hence, we get

$$(3.15) \quad \frac{d}{dt} \|\text{rot}^{(2)} w\|_{L_2(\Omega)}^2 + c_p \nu \|\text{rot}^{(2)} w\|_{L_2(\Omega)}^2 \leq \frac{1}{\nu} \|\bar{h}\|_{L_2(\Omega)}^2,$$

where c_p is the constant from the Poincaré inequality (2.5). Integrating (3.15) with respect to time from $t = kT$ to $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$, we derive

$$(3.16) \quad \begin{aligned} \|\text{rot}^{(2)} w(t)\|_{L_2(\Omega)}^2 &\leq \frac{1}{\nu} e^{-c_p \nu T} \int_{kT}^t \|\bar{h}(t')\|_{L_2(\Omega)}^2 e^{c_p \nu t'} dt' \\ &\quad + \|\text{rot}^{(2)} w(kT)\|_{L_2(\Omega)}^2 \exp(-c_p \nu(t - kT)). \end{aligned}$$

Setting $t = (k+1)T$ in (3.16) yields

$$(3.17) \quad \begin{aligned} \|\text{rot}^{(2)} w((k+1)T)\|_{L_2(\Omega)}^2 &\leq \frac{1}{\nu} \int_{kT}^{(k+1)T} \|\bar{h}(t)\|_{L_2(\Omega)}^2 dt \\ &\quad + \|\text{rot}^{(2)} w(kT)\|_{L_2(\Omega)}^2 \exp(-c_p \nu T). \end{aligned}$$

By iteration we have

$$(3.18) \quad \|\text{rot}^{(2)} w(kT)\|_{L_2(\Omega)}^2 \leq \frac{c_1 A_1^2}{1 - e^{-c_p \nu T}} + \|\text{rot}^{(2)} w(0)\|_{L_2(\Omega)}^2 e^{-c_p \nu kT} \leq A_4^2$$

Hence (3.10) holds. Integrating (3.14) with respect to time from $t = kT$ to $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$, using (3.10) and the Poincaré inequality (2.5) we derive (3.11). This concludes the proof. \square

Remark 3.3. In view of Lemma 2.1 inequalities (3.10) and (3.11) can be expressed in the form

$$(3.19) \quad \|w(kT)\|_{H^1(\Omega)}^2 \leq c_2 A_4^2$$

and

$$(3.20) \quad \|w(t)\|_{H^1(\Omega)}^2 + \nu c_p \int_{kT}^t \|w(t')\|_{H^2(\Omega)}^2 dt' \leq c_2 A_5^2,$$

where c_2 depends on the constant c from (2.2). Proof of existence is standard.

From Lemmas 3.1, 3.2 and Remark 3.3, Theorem 1 follows.

We prove that $w \in C([kT, (k+1)T]; W_\sigma^1(\Omega))$, $\sigma > 3$, $k \in \mathbb{N}_0$. Hence, we want to show that

$$(3.21) \quad \|w(t)\|_{W_\sigma^1(\Omega)} \leq A_6,$$

where A_6 does not depend on time.

The above increasing of regularity is made in [22] by the applying the energy method. This needs much more regularity of data than it is necessary to show (3.21). Moreover, it implies a stronger relation between dissipation and the external force than it is presented in (4.2). Therefore, we follow the regularity increasing technique used in [24].

In this case we have only restriction (4.2). The above mentioned method from [24] is possible because Lemma 3.2 and Remark 3.3 imply that $w \cdot \nabla w \in L_2(kT, (k+1)T; L_\sigma(\Omega))$, $\sigma \in (3, \infty)$ and $k \in \mathbb{N}_0$.

Lemma 3.4. Assume that $w(0) \in B_{\sigma,2}^2(\Omega)$, $\bar{h} \in L_2(kT, (k+1)T; L_\sigma(\Omega))$, $k \in \mathbb{N}_0$, $\sigma > 3$. Then $w \in C(\mathbb{R}_+; W_\sigma^1(\Omega))$, $\sigma > 3$ and (3.21) holds with constant A_6 depending on $\|w(0)\|_{B_{\sigma,2}^2(\Omega)}$ and $\sup_k \int_{kT}^{(k+1)T} \|\bar{h}(t)\|_{L_\sigma(\Omega)} dt$.

Proof. Remark 3.3 implies that $w \cdot \nabla w \in L_2(kT, (k+1)T; L_\sigma(\Omega))$, $\sigma > 3$. In view of the assumptions of the lemma, the theory developed in [12, 13, 19, 20] implies existence of solutions to problem (1.7) such that $w \in W_{\sigma,2}^{2,1}(\Omega \times (kT, (k+1)T))$, $\nabla \bar{\eta} \in L_{\sigma,2}(\Omega \times (kT, (k+1)T))$ and

$$(3.22) \quad \begin{aligned} \|w\|_{W_{\sigma,2}^{2,1}(\Omega \times (kT, (k+1)T))} &\leq c(A_5^2 + \|\bar{h}\|_{L_2(kT, (k+1)T; L_\sigma(\Omega))}) \\ &\quad + \|w(kT)\|_{B_{\sigma,2}^2(\Omega)}, \end{aligned}$$

where c may depend on T . Inequality (3.22) implies (3.21) if we know that $\|w(kT)\|_{B_{\sigma,2}^1(\Omega)}$ is bounded by a constant independent of k . Hence

for $k = 0$, (3.22) implies (3.21). For $k = 1$ and Lemma 2.5 we calculate $\|w(T)\|_{B_{\sigma,2}^1(\Omega)}$ using (3.22) for $k = 0$. Then Lemma 2.8 implies existence of solutions to (1.7) such that

$$\|w\|_{W_{\sigma,2}^{2,1}(\Omega \times (T, 2T))} \leq c,$$

where c depends on $\|w(0)\|_{B_{\sigma,2}^1(\Omega)}$ and $\|\bar{h}\|_{L_2(kT, (k+1)T; L_\sigma(\Omega))}$ for $k = 0, 1$.

To eliminate dependence on $\|w(kT)\|_{B_{\sigma,2}^1(\Omega)}$ in r.h.s. of (3.22) we use a smooth cut-off function $\zeta_k = \zeta_k(t)$ such that $\zeta_k(t) = 0$ for $t \in [kT, kT + \delta/2]$ and $\zeta_k(t) = 1$ for $t \in [kT + \delta, (k+1)T]$, where $\delta < T$. Introducing the quantities

$$w_k = w\zeta_k, \quad \bar{\eta}_k = \bar{\eta}\zeta_k, \quad \bar{h}_k = \bar{h}\zeta_k$$

we see that problem (1.7) takes the form

$$\begin{aligned} (3.23) \quad & w_{k,t} + w \cdot \nabla w_k + \nu \tilde{\text{rot}} \text{rot}^{(2)} w_k + \nabla \bar{\eta}_k = w \dot{\zeta}_k + \bar{h}_k, \\ & \text{div } w_k = 0, \\ & w_k \cdot \bar{n} = 0, \quad \text{rot}^{(2)} w_k = 0 \quad \text{on } S_0, \\ & w_k|_{t=kT} = 0. \end{aligned}$$

In view of Lemmas 2.8 and 3.2 we obtain for solutions to (3.23) the estimate

$$\begin{aligned} (3.24) \quad & \|w_k\|_{W_{\sigma,2}^{2,1}(\Omega \times (kT + \delta/2, (k+1)T))} \\ & \leq c(A_5 + A_5^2 + \|\bar{h}_k\|_{L_{\sigma,2}(\Omega \times (kT + \delta/2, (k+1)T))}) \equiv cA_7, \end{aligned}$$

where in view of the estimate for \bar{h} (see assumptions of Lemma 3.2) we see that A_7 does not depend on k . Then Lemma 2.5 implies

$$(3.25) \quad \|w((k+1)T)\|_{B_{\sigma,2}^1(\Omega)} \leq cA_7.$$

Applying Lemma 2.8 and using (3.25) we obtain

$$\begin{aligned} (3.26) \quad & \|w\|_{W_{\sigma,2}^{2,1}(\Omega \times ((k+1)T, (k+1)T + \delta/2))} \\ & \leq c(A_5 + A_5^2 + \|\bar{h}_k\|_{L_{\sigma,2}(\Omega \times (kT, (k+2)T))}) \leq cA_7 \end{aligned}$$

In view of (3.24) and (3.26) we prove (3.21) with constant A_6 independent of k . This concludes the proof.

Remark 3.5 Remark 3.3 implies that $\|w \cdot \nabla w\|_{L_2(\Omega \times (kT, (k+1)T))} \leq cA_5^2$, so the regularizer technique (see [14, Ch. 4]) gives existence of solutions to problem (1.5) such that $w \in W_2^{2,1}(\Omega \times (kT, (k+1)T))$, $\nabla \eta \in L_2(\Omega \times (kT, (k+1)T))$, $k \in \mathbb{N}_0$ and

$$\|w\|_{W_2^{2,1}(\Omega \times (kT, (k+1)T))} + \|\nabla \eta\|_{L_2(\Omega \times (kT, (k+1)T))} \leq c(A_5 + A_5^2).$$

4. Stability

In this Section we prove stability of two-dimensional solutions. For this purpose we examine problem (1.9). First we show the L_2 -stability.

Lemma 4.1. *Let the assumptions of Lemma 3.2 be satisfied. Assume that*

$$(4.1) \quad \begin{aligned} B_1^2 &= \sup_k \int_{kT}^{(k+1)T} \|g(t)\|_{L_2(D)}^2 dt, \quad B_2^2 = \frac{c}{\nu} \exp\left(\frac{c}{\nu} A_5^2\right) B_1^2, \\ B_3^2 &= \frac{B_2^2}{1 - \exp\left(-\frac{\nu}{2} T\right)} + \|u(0)\|_{L_2(D)}^2. \end{aligned}$$

$$(4.2) \quad -\frac{\nu}{2} T + \frac{c}{\nu} A_5^2 \leq 0.$$

Then the following estimates for solutions to (1.9) hold

$$(4.3) \quad \|u(kT)\|_{L_2(D)}^2 \leq B_3^2$$

$$(4.4) \quad \|u(t)\|_{L_2(D)}^2 \leq \exp\left(\frac{c}{\nu} A_5^2\right) \frac{c}{\nu} B_1^2 + B_3^2 \equiv B_4^2.$$

Proof. Multiplying (1.9)₁ by u , integrating over D and using the boundary conditions yield

$$(4.5) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L_2(D)}^2 + \nu \|\operatorname{rot} u\|_{L_2(D)}^2 = - \int_D u \cdot \nabla w \cdot u dx + \int_D g \cdot u dx$$

Applying the Hölder and the Young inequalities to the r.h.s. of (4.5) and using Lemma 2.1 we obtain

$$(4.6) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L_2(D)}^2 + \frac{\nu}{2} \|\operatorname{rot} u\|_{L_2(D)}^2 &\leq \frac{c_e}{\nu} \|\nabla w\|_{L_3(D)}^2 \|u\|_{L_2(D)}^2 \\ &\quad + \frac{1}{\nu} \|g\|_{L_2(D)}^2, \end{aligned}$$

where c_e appears in (2.8). Applying again Lemma 2.1 we have

$$(4.7) \quad \begin{aligned} \frac{d}{dt} \|u\|_{L_2(D)}^2 + \nu \|u\|_{L_2(D)}^2 &\leq \frac{c}{\nu} \|\nabla w\|_{L_3(D)}^2 \|u\|_{L_2(D)}^2 \\ &\quad + \frac{c}{\nu} \|g\|_{L_2(D)}^2. \end{aligned}$$

Inequality (4.7) implies

$$\begin{aligned}
(4.8) \quad & \frac{d}{dt} \left[\|u\|_{L_2(D)}^2 \exp \left(\nu t - \frac{c}{\nu} \int_{kT}^t \|\nabla w(t')\|_{L_3(D)}^2 dt' \right) \right] \\
& \leq \frac{c}{\nu} \|g\|_{L_2(D)}^2 \exp \left(\nu t - \frac{c}{\nu} \int_{kT}^t \|\nabla w(t')\|_{L_3(D)}^2 dt' \right).
\end{aligned}$$

Integrating (4.8) with respect to time from $t = kT$ to $t \in (kT, (k+1)T]$ gives

$$\begin{aligned}
(4.9) \quad & \|u(t)\|_{L_2(D)}^2 \leq \exp \left(-\nu t + \frac{c}{\nu} \int_{kT}^t \|\nabla w(t')\|_{L_3(D)}^2 dt' \right) \\
& \cdot \frac{c}{\nu} \int_{kT}^t \|g(t')\|_{L_2(D)}^2 \exp(\nu t') dt' \\
& + \exp \left(-\nu(t - kT) + \frac{c}{\nu} \int_{kT}^t \|\nabla w(t')\|_{L_3(D)}^2 dt' \right) \|u(kT)\|_{L_2(D)}^2.
\end{aligned}$$

Setting $t = (k+1)T$ and employing (3.11) we have

$$\begin{aligned}
(4.10) \quad & \|u((k+1)T)\|_{L_2(D)}^2 \leq \exp \left(\frac{c}{\nu} A_5^2 \right) \frac{c}{\nu} \int_{kT}^{(k+1)T} \|g(t')\|_{L_2(D)}^2 dt' \\
& + \exp \left(-\nu T + \frac{c}{\nu} A_5^2 \right) \|u(kT)\|_{L_2(D)}^2.
\end{aligned}$$

In view of assumptions (4.1) inequality (4.10) takes the form

$$(4.11) \quad \|u((k+1)T)\|_{L_2(D)}^2 \leq B_2^2 + \exp \left(-\frac{\nu}{2} T \right) \|u(kT)\|_{L_2(D)}^2.$$

By iteration we have

$$(4.12) \quad \|u(kT)\|_{L_2(D)}^2 \leq \frac{B_2^2}{1 - \exp \left(-\frac{\nu}{2} T \right)} + \exp \left(-\frac{\nu}{2} kT \right) \|u(0)\|_{L_2(D)}^2 \leq B_3^2,$$

so (4.3) holds. Applying (4.12), (4.2) and (4.3) in (4.9) yields (4.4). This concludes the proof.

Finally, we prove stability omitting the strong restriction (4.2).

Lemma 4.2. *Let $w \in L_2(kT, (k+1)T; W_{2+}^1(\Omega))$, $g \in C(kT, (k+1)T; L_2(D))$, $k \in \mathbb{N}_0$. Let c_* be a constant such that $c_* \in (0, \nu]$. Let γ_* be so small that $\frac{c_*}{2} \leq \nu - \frac{c_0}{\nu^3} \gamma_*^4$, where c_0 is the constant from (4.20). Let $\gamma \leq \gamma_*$. Let*

$$(4.13) \quad \begin{aligned} \|u(0)\|_{H^1(\Omega)}^2 &\leq \gamma \\ G^2(t) &= \frac{c}{\nu} \|g(t)\|_{L_2(D)}^2 \leq \frac{c_*}{4} \gamma. \end{aligned}$$

Then

$$(4.14) \quad \|u(t)\|_{H^1(\Omega)}^2 \leq \gamma \quad \text{for } t \in \mathbb{R}_+.$$

Proof. Multiplying (1.9) by $\text{rot}^2 u$, integrating over D and by parts yields

$$(4.15) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\text{rot} u\|_{L_2}^2 + \nu \|\text{rot}^2 u\|_{L_2}^2 &\leq - \int_D u \cdot \nabla u \cdot \text{rot}^2 u dx \\ &\quad - \int_D w \cdot \nabla u \cdot \text{rot}^2 u dx - \int_D u \cdot \nabla w \cdot \text{rot}^2 u dx + \int_D g \text{rot}^2 u dx. \end{aligned}$$

The first term on the r.h.s. of (4.15) equals

$$\begin{aligned} & - \int_{\Omega} \text{rot}(u \cdot \nabla u) \text{rot} u dx \\ &= - \int_{\Omega} u \cdot \nabla \text{rot} u \text{rot} u dx - \int_{\Omega} \varepsilon_{kij} u_{l,x_j} \partial_{x_l} u_i (\text{rot} u)_k dx \equiv I_1, \end{aligned}$$

where ε_{kij} is the antisymmetric Ricci tensor and summation is performed over all repeated indices. Since the first term in I_1 vanishes in view of the boundary conditions, we have

$$|I_1| \leq c \|u_x\|_{L_3(D)}^3.$$

Applying the Hölder and the Young inequalities to the other terms on the r.h.s. of (4.15), we obtain

$$(4.16) \quad \begin{aligned} \frac{d}{dt} \|\text{rot} u\|_{L_2(D)}^2 + \nu \|\text{rot}^2 u\|_{L_2(D)}^2 &\leq c \|u_x\|_{L_3(D)}^3 + \frac{c}{\nu} \|w \cdot \nabla u\|_{L_2(D)}^2 \\ &\quad + \frac{c}{\nu} \|u \cdot \nabla w\|_{L_2(D)}^2 + \frac{c}{\nu} \|g\|_{L_2(D)}^2. \end{aligned}$$

In view of the interpolation inequality

$$\|u_x\|_{L_3(D)} \leq c \|u_x\|_{H^1(D)}^{1/2} \|u_x\|_{L_2(D)}^{1/2}$$

the first term on the r.h.s. of (4.16) is bounded by

$$c \|u_x\|_{H^1(D)}^{3/2} \|u_x\|_{L_2(D)}^{3/2} \leq \frac{\varepsilon_1^{4/3}}{4/3} \|u_x\|_{H^1(D)}^2 + \frac{c}{4\varepsilon_1^4} \|u_x\|_{L_2(D)}^6 \equiv I_2.$$

Using Lemma 2.3 we have

$$\|u_x\|_{L_2(D)} \leq \|u\|_{H^1(D)} \leq c \|\operatorname{rot} u\|_{L_2(D)}$$

and Lemma 2.4 gives

$$(4.17) \quad \|u_x\|_{H^1(D)} \leq c \|\operatorname{rot} u\|_{H^1(D)} \leq c \|\operatorname{rot}^2 u\|_{L_2(D)}.$$

Setting $\frac{\varepsilon_1^{4/3}}{4/3} = \frac{\nu}{4c}$ we obtain that

$$I_2 \leq \frac{\nu}{4} \|\operatorname{rot}^2 u\|_{L_2(D)}^2 + \frac{c}{\nu^3} \|\operatorname{rot} u\|_{L_2(D)}^6.$$

The second term on the r.h.s. (4.16) is bounded by

$$\frac{c}{\nu} \|w\|_{L_\infty(D)}^2 \|\nabla u\|_{L_2(D)}^2 \leq \frac{c}{\nu} \|w\|_{W_{2^+}^1(\Omega)}^2 \|\nabla u\|_{L_2(D)}^2,$$

where $2^+ > 2$ but is very close to 2.

Finally, the third term on the r.h.s. of (4.16) is estimated by

$$\frac{c}{\nu} \|u\|_{L_6(D)}^2 \|\nabla w\|_{L_3(\Omega)}^2 \leq \frac{c}{\nu} \|u\|_{H^1(D)}^2 \|\nabla w\|_{L_3(\Omega)}^2.$$

Employing the above estimates in (4.16) yields the inequality

$$(4.18) \quad \begin{aligned} \frac{d}{dt} \|\operatorname{rot} u\|_{L_2(D)}^2 + \nu \|\operatorname{rot}^2 u\|_{L_2(D)}^2 &\leq \frac{c}{\nu^3} \|\operatorname{rot} u\|_{L_2(D)}^6 \\ &+ \frac{c}{\nu} \|w\|_{W_{2^+}^1(\Omega)}^2 \|u\|_{H^1(D)}^2 + \frac{c}{\nu} \|g\|_{L_2(D)}^2. \end{aligned}$$

Let us introduce the quantities

$$(4.19) \quad \begin{aligned} X(t) &= \|\operatorname{rot} u(t)\|_{L_2(D)}, & Y(t) &= \|\operatorname{rot}^2 u(t)\|_{L_2(D)}, \\ G^2(t) &= \frac{c}{\nu} \|g(t)\|_{L_2(D)}^2, & A^2(t) &= \frac{c}{\nu} \|w\|_{W_{2^+}^1(\Omega)}^2 \end{aligned}$$

To prove the lemma we need to know that $G(t) \in C([kT, (k+1)T])$, $A^2 \in L_1(kT, (k+1)T)$, $k \in \mathbb{N}_0$. By the assumptions of the lemma we know that $\|g(t)\|_{L_2(D)} \in C([kT, (k+1)T])$. Similarly, Lemma 4.1 implies that $\|u(t)\|_{L_2(D)} \in C([kT, (k+1)T])$. Moreover, Lemma 3.4 shows that $w \in C([kT, (k+1)T]; W_{2+}^1(\Omega))$ for all $k \in \mathbb{N}_0$.

The aim of this lemma is to show that $\|u(t)\|_{H^1(D)}$ is as small as $\|u(0)\|_{H^1(D)}$ in each time interval $[kT, (k+1)T]$, $k \in \mathbb{N}_0$, separately. Therefore, in view of notation (4.19) and with the above shown properties of $G(t)$, we express (4.18) in the short form

$$(4.20) \quad \frac{d}{dt}X^2 + \nu Y^2 \leq \frac{c_0}{\nu^3}X^6 + A^2X^2 + G^2.$$

Since $X \leq Y$ we have

$$(4.21) \quad \frac{d}{dt}X^2 \leq -X^2 \left(\nu - \frac{c_0}{\nu^3}X^4 \right) + A^2X^2 + G^2.$$

Let $\gamma \in (0, \gamma_*]$, where γ_* is so small that

$$(4.22) \quad \nu - \frac{c_0}{\nu^3}\gamma^4 \geq \frac{c_*}{2}, \quad 0 < c_* \leq \nu.$$

Since the coefficients of (4.21) depend on the two-dimensional solutions determined step by step in time, we consider (4.21) in the interval $[kT, (k+1)T]$, $k \in \mathbb{N}_0$, with the assumptions

$$(4.23) \quad X^2(kT) \leq \gamma, \quad G^2(t) \leq c_* \frac{\gamma}{4}, \quad t \in [kT, (k+1)T].$$

Let us introduce the quantity

$$Z^2(t) = \exp \left(- \int_{kT}^t A^2(t') dt' \right) X^2(t), \quad t \in [kT, (k+1)T].$$

Then (4.21) takes the form

$$(4.24) \quad \frac{d}{dt}Z^2 \leq - \left(\nu - \frac{c_0}{\nu^3}X^4 \right) Z^2 + \bar{G}^2,$$

where $\bar{G}^2 = G^2 \exp \left(- \int_{kT}^t A^2(t') dt' \right)$.

Suppose that

$$\begin{aligned} t_* &= \inf \{ t \in (kT, (k+1)T] : X^2(t) > \gamma \} \\ &= \inf \left\{ t \in (kT, (k+1)T] : Z^2(t) > \gamma \exp \left(- \int_{kT}^t A^2(t') dt' \right) \right\} > kT. \end{aligned}$$

By (4.22) for $t \in (0, t_*]$ inequality (4.24) takes the form

$$(4.25) \quad \frac{d}{dt} Z^2 \leq -\frac{c_*}{2} Z^2 + \bar{G}^2(t)$$

Clearly, we have

$$(4.26) \quad Z^2(t_*) = \gamma \exp \left(- \int_{kT}^{t_*} A^2(t') dt' \right) \quad \text{and} \quad Z^2(t) > \gamma \exp \left(- \int_{kT}^t A^2(t') dt' \right)$$

for $t > t_*$. But (4.23) and (4.25) yield

$$\frac{d}{dt} Z^2|_{t=t_*} \leq c_* \left(-\frac{\gamma}{2} + \frac{\gamma}{4} \right) \exp \left(- \int_{kT}^{t_*} A^2(t') dt' \right) < 0$$

so it contradicts to (4.26). Hence $X^2((k+1)T) < \gamma$. Then induction proves the lemma. \square

Remark 4.3 Let the assumptions of Remark 3.3 and Lemma 4.2 hold. Then the regularizer technique (see [14, Ch 4]) gives existence of solutions to problem (1.9) such that $u \in W_2^{2,1}(D \times (kT, (k+1)T))$, $\nabla q \in L_2(D \times (kT, (k+1)T))$, $k \in \mathbb{N}_0$, and

$$\|u\|_{W_2^{2,1}(D \times (kT, (k+1)T))} + \|\nabla q\|_{L_2(D \times (kT, (k+1)T))} \leq c\gamma[A_5 + \gamma + 1].$$

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